THEOREMS ON INTERACTION OF PARTS OF A MECHANICAL SYSTEM

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For the solution of problems of dynamics the methods which do not raise the question of determining reaction forces of constraints are of greatest interest. As a rule, reactions of constrainst in the systems are not known beforehand and their general properties follow from the properties of constraints. The basic idea of these methods is to express the properties of constraints through properties of possible displacements. Considering mechanical motions as coordinate transformations, it is possible to establish some correspondence between possible displacements and these transformations. In other words, it is possible to separate from the class of possible displacements all those possible displacements which have the properties of some transformations. Then, for such possible displacements, these transformations yield some properties of the mechanical system which, under certain restraints placed on forces, reduce to the existence of first integrals of equations of motion of the system.

From this point of view in this paper a certain generalization of the first two fundamental theorems of dynamics of a mechanical system consisting of an arbitrary number of material points $m_1, m_2, \ldots$, with smooth holonomic constraints imposed on it, is examined. Theorems on motion of center of mass of the system and on angular momenta of the system are considered.

1. Let the mechanical system Λ consisting of an arbitrary number of material points $m_1, m_2, \ldots$, be divisible into parts (1) and (2) the properties of which are characterized by applied constraints, acting forces and reactions in the following manner. We introduce two systems of Cartesian coordinates xyz and $x'y'z'$ which always remain parallel with respect to a stationary system of coordinates $x_1y_1z_1$.

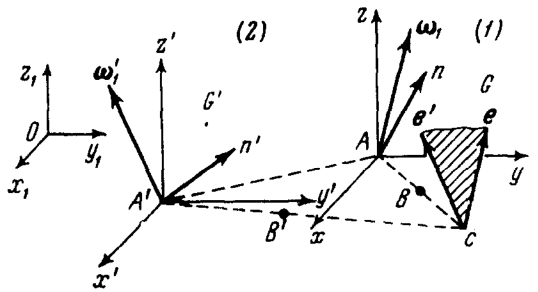
The origins of non-stationary systems of coordinates $Axyz$ and $A'x'y'z'$, are located at certain points A and A' respectively, with reference to systems (1) and (2).

Location of points A and A' with reference to the stationary axes $x_1y_1z_1$ is determined by coordinates: α, β, γ and α', β', γ' which are related to coordinates α°, β°, γ° and $\alpha^{\circ\prime}$, $\beta^{\circ\prime}$, $\gamma^{\circ\prime}$ of the centers of gravity G and G' of the system (1) and (2) with respect to the same axes by the following relationships

$$\begin{aligned} \alpha^{\circ} &= \lambda\alpha + \alpha_0, & \beta^{\circ} &= \lambda\beta + \beta_0, & \gamma^{\circ} &= \lambda\gamma + \gamma_0 \\ \alpha^{\circ\prime} &= \lambda'\alpha' + \alpha_0', & \beta^{\circ\prime} &= \lambda'\beta' + \beta_0', & \gamma^{\circ\prime} &= \lambda'\gamma' + \gamma_0' \end{aligned} \tag{1.1}$$

Here α_0, β_0, γ_0, α_0', β_0', and γ_0' are arbitrary constants.

Velocities of points A and A' are parallel to the velocities of centres of gravity G of system (1) and G' of system (2) respectively. Let smooth constraints applied to the system be such, that, one can introduce into systems (1) and (2) possible helical displacements without changing the relative distribution of points within each part of the system.



These helical displacements of systems (1) and (2) can be represented (according to Hamilton) by two bivectors the principal parts of which (rotations) are oriented along two straight lines ω_1 and ω_1', passing through the moving points A and A'.

Moments of bivectors (translational displacements) also pass through the points A and A' respectively in the directions of the straight lines n and n' (fig.).

For system (1) displacements of an arbitrary points of a rigid body are obtained in projections on stationary axes according to Euler's equations by the cyclic permutation S of four groups of letters (x_1, y_1, z_1; α, β, γ; π_1, σ_1, ρ_1; x, y, z) according to equation

$$\delta x_1 = \delta\alpha + \sigma_1 z - \rho_1 y \qquad (S) \tag{1.2}$$

Here π_1, σ_1, and ρ_1 are the projections on stationary axes of instantaneous infinitely small rotation of the solid body (1); $\delta\alpha$, $\delta\beta$, and $\delta\gamma$ are projections of the displacement of point A along the straight line n. Equations (1.2) can be represented in the form

$$\delta x_1 = \lambda_1 + \sigma_1 z_1 - \rho_1 y_1 \quad (S) \tag{1.3}$$

Here the cyclic permutation includes the group of letters λ_1, μ_1 and ν_1, the magnitudes of which are given by

$$\lambda_1 = \delta\alpha - \sigma_1\gamma + \rho_1\beta \quad (S) \tag{1.4}$$

Displacements (1.3) of the rigid body consist of rotation around an axis referred to the origin of the stationary system of coordinates, rotational components being π_1, σ_1, and ρ_1 and translation with components λ_1, μ_1, and ν_1. The latter, according to (1.4), consists of two parts: the first part is a translation which coincides with the displacement of the point A (α, β, γ); the second part coincides with displacement of a point of the solid body located at the origin of the stationary system of coordinates, as a result of rotation of the solid body around an axis with respect to point A. Projections of the components of this rotation on stationary axes will be the same; π_1, σ_1, and ρ_1.

Equations (1.2) express displacements of a solid body consisting of a translation and rotation if the translation which coincides with displacement of point A is characterized by a free vector with components $\delta\alpha$, $\delta\beta$, and $\delta\gamma$ and if the rotation around the axis which goes through the origin of non-stationary system of axes A, is characterized by a sliding vector with components π_1, σ_1, and ρ_1.

Equations (1.3) are obtained as a result of bringing these two vectors (the free vector and the sliding vector) to the origin of the stationary system of coordinates.

Freedom in the choice of the stationary system of coordinates makes it possible to orient the axis z_1 parallel to the axis of rotation and to place it so, that for the same orientation of x and y axes $\delta x = \delta y = 0$.

Then, in such a stationary system of coordinate axes $x_1^* y_1^* z_1^*$ we shall have $\pi_1 = \sigma_1 = 0$, and Equations for displacements will take the form

$$\delta x_1^* = - y_1^* \rho_1, \qquad \delta y_1^* = x_1^* \rho_1, \qquad \delta z_1^* = \nu_1 \tag{1.5}$$

The displacement will be an infinitely small helical displacement of the rigid body, consisting of a positive displacement along the z^*_1 axis by the quantity ν_1 and of a rotation around this axis by the quantity ρ_1 [1].

Designating all quantities related to the coordinate system $x'y'z'$ by the index 'prime above' and introducing the free and sliding vectors which characterize the displacements of system (2) as a rigid body, we obtain

$$\delta x_1' = \delta\alpha' + \sigma_1' z' - \rho_1' y' \qquad (S') \tag{1.6}$$

The remaining equations are obtained by cyclic permutation of letters

$$S' \ (x_1', y_1', z_1';\ \alpha', \beta', \gamma';\ \pi_1', \sigma_1', \rho_1';\ x', y', z')$$

Here π_1', σ_1', and ρ_1' are the projections on stationary axes of an instantaneous infinitely small rotation of solid body (2); $\delta\alpha$, $\delta\beta$, and $\delta\gamma$ are the projections of translation of the point A' along the straight line n'.

Equations (1.2) and (1.6) will correspond to infinitely small helical displacements of solid bodies (1) and (2).

Let us draw a plane π parallel to the vectors ω_1 and ω_1' and let the plane pass through the point C located at the intersection of two straight lines with constant direction AB and $A'B'$. Points B and B' have fixed locations in coordinate systems xyz and $x'y'z'$ respectively.

We shall construct from point C two unit vectors $\mathbf{e}$ and $\mathbf{e}'$ in the plane π and parallel to the vectors ω_1 and ω_1' respectively.

Let a, b and c be the coordinates of point B in the system of axes xyz; let a', b' and c' be the coordinates of point B' in the system of axes $x'y'z'$; and let l_0, m_0, n_0 and l_0', m_0', n_0' be the direction cosines of the unit vectors $\mathbf{e}$ and $\mathbf{e}'$.

Let us now designate by μ ($\mu \neq \mathrm{const}$) the ratio of sections $AC : AB$; by μ' ($\mu' \neq \mathrm{const}$) the ratio of sections $A'C : A'B'$ (see figure); by X, Y and Z the projections of external forces acting on the points of the system Λ; by M the sum of masses of system (1); by M' the sum of masses of system (2) and by $\mathbf{A}$ and $\mathbf{A}'$ vectors, which coincide with sections AB and $A'B'$ respectively.

Coordinates of the center of gravity G of system (1) relative to the axes xyz and coordinates of center of gravity G' of system (2) relative to the axes $x'y'z'$ will respectively be

$$(\lambda - 1)\alpha + \alpha_0, \qquad (\lambda - 1)\beta + \beta_0, \qquad (\lambda - 1)\gamma + \gamma_0$$
$$(\lambda' - 1)\alpha' + \alpha_0', \qquad (\lambda' - 1)\beta' + \beta_0', \qquad (\lambda' - 1)\gamma' + \gamma_0'$$

In the moving system of coordinates connected with points A and A' we have

$$\begin{gathered}\Sigma mx = M\,[(\lambda - 1)\alpha + \alpha_0], \quad \Sigma my = M\,[(\lambda - 1)\beta + \beta_0] \\ \Sigma mz = M\,[(\lambda - 1)\gamma + \gamma_0] \\ \Sigma' mx' = M'\,[(\lambda' - 1)\alpha' + \alpha_0'], \quad \Sigma' my' = M'\,[(\lambda' - 1)\beta' + \beta_0'] \\ \Sigma' mz' = M'\,[(\lambda' - 1)\gamma' + \gamma_0']\end{gathered} \tag{1.7}$$

Here and in the following we use the notation: Σ is the sum over the points of the system (1); Σ' is the sum over the points of the system (2).

Properties of constraints placed on system Λ, are determined on the following premises:

1°. The constraints are smooth and permit the rotation of system (1) about the straight line ω_1 and a positive displacement along the straight line n, rotation of system (2) about the straight line ω_1' and positive a displacement along the straight line n' in the manner of rigid bodies.

2°. Moving straight lines ω_1 and ω_1' passing through A and A' respectively, have fixed directions.

3°. Straight lines n and n' are perpendicular to the planes passing through $\mathbf{A}'$, $\boldsymbol{\omega}_1'$ and $\mathbf{A}$, $\boldsymbol{\omega}_1$ respectively.

We select possible displacements $\delta\mathbf{l}$ $(\delta\alpha, \delta\beta, \delta\gamma)$ along the straight line n and $\delta\mathbf{l}'$ $(\delta\alpha', \delta\beta', \delta\gamma')$ along the straight line n' in such a manner that the following equations are satisfied.

$$\delta\mathbf{l} = \chi'\mu'\mathbf{e}' \times \mathbf{A}', \qquad \delta\mathbf{l}' = \chi\mu\mathbf{e} \times \mathbf{A} \tag{1.8}$$

Here X' and X are constant coefficients of proportionality.

The magnitudes of possible rotations ω_1 and ω_1' are selected so as to satisfy the following relations continuously

$$\begin{gathered}\omega_1' = K\omega_1 \qquad (K=\text{const}) \\ \boldsymbol{\omega}_1 = \omega_1\mathbf{e}, \qquad \pi_1 = \omega_1 l_0, \qquad \sigma_1 = \omega_1 m_0, \qquad \rho_1 = \omega_1 n_0 \\ \boldsymbol{\omega}_1' = \omega_1'\mathbf{e}', \qquad \pi_1' = \omega_1' l_0', \qquad \sigma_1' = \omega_1' m_0', \qquad \rho_1' = \omega_1' n_0\end{gathered} \tag{1.9}$$

Let possible helical displacements of rigid bodies (1) and (2) be selected such that when (1.7) and (1.8) are satisfied the following equations apply

$$\chi'\mu' = \varkappa'\omega_1', \qquad \chi\mu = \varkappa\omega_1 \qquad (\varkappa', \varkappa = \text{const}) \tag{1.10}$$

Forces acting on system Λ have the following properties:

4°. Systems (1) and (2) are under the influence of arbitrary internal forces.

5°. If external forces acting on system (1) are summed under the assumption of invariability of the system, they are reduced at the point A to force $\mathbf{F}$, which is located in a plane through A and is parallel to the straight lines $A'C$ and ω_1', and to the couple with the moment $\mathbf{M}^\circ$.

Also the following is true

$$\mathbf{F}\boldsymbol{\omega}_1'\mathbf{A}'=0 \qquad (1.11)$$

6°. When external forces acting on system (2) are reduced at the point A' under the same assumptions, they result in forces $\mathbf{F}'$ and a couple with the moment $\mathbf{M}^{\circ\prime}$. The force $\mathbf{F}'$ is located in a plane through A' parallel to the straight lines AC and ω_1. The following relationship is applicable

$$\mathbf{F}'\boldsymbol{\omega}_1\mathbf{A}=0 \qquad (1.12)$$

7°. Projection of the moment of the first couple on the direction ω_1, multiplied by ω_1 added to the projection of the moment of the second couple on the direction ω_1', multiplied by ω_1', gives zero

$$\boldsymbol{\omega}_1\cdot\mathbf{M}^\circ+\boldsymbol{\omega}_1'\cdot\mathbf{M}^{\circ\prime}=0 \qquad (1.13)$$

This would take place for example when moments $\mathbf{M}^\circ$ and $\mathbf{M}^{\circ\prime}$ are perpendicular to $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_1'$.

Let the forces of the system (1) which act on system (2) be reduced at the point A to reaction $\mathbf{R}$ (R_x, R_y, R_z) and a couple with a moment $\mathbf{H}$ the projections of which on the axes of coordinates xyz will be H_x, H_y, and H_z, at the point A' to reaction R and a couple with a moment $\mathbf{H}'$ which has projections $H'_{x'}$, $H'_{y'}$, and $H'_{z'}$, on the axes $x'y'z'$, and at point C to reaction $\mathbf{R}$ and a couple the moment of which will be $\mathbf{W}$

$$\mathbf{W}=\mathbf{H}-\mu\mathbf{A}\times\mathbf{R}=\mathbf{H}'-\mu'\mathbf{A}'\times\mathbf{R} \qquad (1.14)$$

In the reduction of reaction forces the mechanical system is assumed to be invariable.

8°. Forces of action of the system (1) on system (2) are such that the moment $\mathbf{W}$ is perpendicular to the plane π, i.e. to vectors $\mathbf{e}$ and $\mathbf{e}'$

$$\mathbf{W}\cdot\mathbf{e}=0, \qquad \mathbf{W}\cdot\mathbf{e}'=0 \qquad (1.15)$$

Instead of the requirement indicated, it could be assumed that the moment $\mathbf{W}$ is located in the plane perpendicular to a straight line which passes through the ends of the vectors $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_1'$, constructed at the point C

$$\mathbf{W}\cdot(\boldsymbol{\omega}_1-\boldsymbol{\omega}_1')=0$$

9°. In reducing the indicated forces to the point C the following equation is satisfied

$$\boldsymbol{\omega}_1(\varkappa-\mu)\,\mathbf{R}\mathbf{e}\mathbf{A}=\boldsymbol{\omega}_1'(\varkappa'-\mu')\,\mathbf{R}\mathbf{e}'\mathbf{A}' \qquad (1.16)$$

This equation, could for example be satisfied if reaction **R** constructed at the point C was located at the intersection of planes passing through **e**, **A** and **e**$'$, **A**$'$.

Let us apply the fundamental principle of dynamics of systems of material points

$$\sum\left\{\left(m\frac{d^2x_1}{dt^2}-X\right)\delta x_1+\left(m\frac{d^2y_1}{dt^2}-Y\right)\delta y_1+\left(m\frac{d^2z_1}{dt^2}-Z\right)\delta z_1\right\}=0$$

to the systems (1) and (2) assuming, that possible displacements are determined by Equations (1.2) and (1.6). In accordance with Equations (1.1), and (1.7) and Equations for transition from the stationary system of coordinates to the non-stationay one, we have, for the system (1)

$$\begin{gathered}\delta\alpha\frac{d}{dt}\lambda M\frac{d\alpha}{dt}+\delta\beta\frac{d}{dt}\lambda M\frac{d\beta}{dt}+\delta\gamma\frac{d}{dt}\lambda M\frac{d\gamma}{dt}+\pi_1\frac{dS_1}{dt}+\sigma_1\frac{dS_2}{dt}+\rho_1\frac{dS_3}{dt}=\\=\delta\alpha(\Sigma X-R_x)+\delta\beta(\Sigma Y-R_y)+\delta\gamma(\Sigma Z-R_z)+\pi_1[\Sigma(yZ-zY)-H_x]+\\+\sigma_1[\Sigma(zX-xZ)-H_y]+\rho_1[\Sigma(xY-yX)-H_z]\end{gathered}\tag{1.17}$$

Here

$$\begin{aligned}S_1&=(\lambda-1)M\left(\beta\frac{d\gamma}{dt}-\gamma\frac{d\beta}{dt}\right)+M\left(\beta_0\frac{d\gamma}{dt}-\gamma_0\frac{d\beta}{dt}\right)+\Sigma m\left(y\frac{dz}{dt}-z\frac{dy}{dt}\right)\\S_2&=(\lambda-1)M\left(\gamma\frac{d\alpha}{dt}-\alpha\frac{d\gamma}{dt}\right)+M\left(\gamma_0\frac{d\alpha}{dt}-\alpha_0\frac{d\gamma}{dt}\right)+\Sigma m\left(z\frac{dx}{dt}-x\frac{dz}{dt}\right)\\S_3&=(\lambda-1)M\left(\alpha\frac{d\beta}{dt}-\beta\frac{d\alpha}{dt}\right)+M\left(\alpha_0\frac{d\beta}{dt}-\beta_0\frac{d\alpha}{dt}\right)+\Sigma m\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)\end{aligned}\tag{1.18}$$

designate the sums of angular momenta with respect to the axes x, y, z of the system (1).

Using the above principle, we have, for the system (2) in the same manner

$$\begin{gathered}\delta\alpha'\frac{d}{dt}\lambda'M'\frac{d\alpha'}{dt}+\delta\beta'\frac{d}{dt}\lambda'M'\frac{d\beta'}{dt}+\delta\gamma'\frac{d}{dt}\lambda'M'\frac{d\gamma'}{dt}+\pi_1'\frac{dS_1'}{dt}+\sigma_1'\frac{dS_2'}{dt}+\\+\rho_1'\frac{dS_3'}{dt}=\delta\alpha'(\Sigma'X+R_x)+\delta\beta'(\Sigma'Y+R_y)+\delta\gamma'(\Sigma'Z+R_z)+\\+\pi_1'[\Sigma'(y'Z-z'Y)+H_{x'}']+\sigma_1'[\Sigma'(z'X-x'Z)+H_{y'}']+\rho_1'[\Sigma'(x'Y-y'X)+H_{z'}']\end{gathered}\tag{1.19}$$

Here S_1', S_2', and S_3' have the same expressions as (1.18) (if the index 'prime above' is added to all quantities), and designate the sums of angular momenta with respect to the axes x', y', and z' of the system (2).

Adding Equations (1.17) and (1.19) we obtain zero in the right hand side by virtue of Equations (1.8) and (1.9) and conditions (1.11 to 1.16). The left hand side gives the first integral

$$\begin{gathered}K\varkappa'\lambda M\left\{l_0'\left(b'\frac{d\gamma}{dt}-c'\frac{d\beta}{dt}\right)+m_0'\left(c'\frac{d\alpha}{dt}-a'\frac{d\gamma}{dt}\right)+n_0'\left(a'\frac{d\beta}{dt}-b'\frac{d\alpha}{dt}\right)\right\}+\\+\varkappa\lambda'M'\left\{l_0\left(b\frac{d\gamma'}{dt}-c\frac{d\beta'}{dt}\right)+m_0\left(c\frac{d\alpha'}{dt}-a\frac{d\gamma'}{dt}\right)+n_0\left(a\frac{d\beta'}{dt}-b\frac{d\alpha'}{dt}\right)\right\}+\\+l_0S_1+m_0S_2+n_0S_3+K(l_0'S_1'+m_0'S_2'+n_0'S_3')=\text{const}\end{gathered}\tag{1.20}$$

Equations (1.17) and (1.19) make it possible to obtain theorems of interaction between the parts (1) and (2) of the system Λ. The first integral (1.20) is the mathematical expression for these theorems. It characterizes the mutual interaction of two parts of the system under the conditions earlier mentioned on the constraints, external forces and forces of interaction of both parts. These theorems will constitute a certain generalization of the first two basic theorems of dynamics and will have different formulation for individual cases of action of momentum and principal moment of one part of the system on the momentum and principal moment of another part of the system. Examples of this are presented in section (2).

There is no necessity here to give complete formulations of these theorems. They are understandable from methods of derivation of general theorems of dynamics of systems of material particles.

Their brief contents is as follows: For a mechanical system, which consists of an arbitrary number of material points, with constraints placed on it which permit displacements with characteristics indicated in the points 1° to 3° and which is under the action of forces with characteristics indicated in the points 4° to 9°, integral (1.20) is applicable. This integral will be a certain generalization of integrals of the first two general theorems of dynamics, the theorems about motion of the center of mass of the system and about the angular momentum of the system.

2. Integrals of the first two general theorems of dynamics and generalized integrals of Chaplygin [3] are obtained as special cases of the integral (1.20).

(*a*) We assume that α', β', and γ' are constant, $l_0 = m_0 = 0$, $n = 1$, $K = 0$, and $a = b = c = 0$.

In accordance with (1.8), (1.9) and (1.10) this is equivalent to the situation where system (2) is absent while system (1) can rotate without change in configuration arround the straight line Az.

External forces acting in this case on the system (1) result in a moment which is with respect to the straight line Az, equal to zero. Integral (1.20) gives $S_z = \text{const}$, i.e. the generalized integral of the Chaplygin areas (section 1 of [3]).

(*b*) We assume that $\varkappa' = \varkappa = \chi' = \chi = 0$, $l_0 = m_0 = l_0' = m_0' = 0$, and $n_0 = n_0' = 1$.

In accordance with (1.8), (1.9) and (1.10) this is equivalent to the situation where the parts (1) and (2) of the system, have the properties of the point a of system (1) with reference to the constraints, external forces, centers A and A' and the axes Az and $A'z'$. It follows from Expression (1.16) that for arbitrary $\mathbf{R}$ possible rotations $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_1'$ must be such that the moments of sliding vectors $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_1'$ relative to the point C are equal.

Point C must necessarily lie in the plane passing through $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_1'$. Without sacrificing generality we can assume it to be on the straight line $A'A$.

The constant K will be equal to the ratio of the segments $CA : CA'$.

Interaction forces between the systems (1) and (2) have, according to (1.14) and (1.15), a moment equal to zero with respect to a straight line which is parallel to z and passes through the point C. This gives

$$H_z : H'_{z'} = K$$

Integral (1.20) becomes equal to

$$S_3 + KS_3' = \text{const}$$

i.e. the integral of Chaplygin (integral (4), section 2 of [3]) is obtained.

(*c*) We assume that $\chi' = 0$, $K = 0$, $l_0 = m_0 = 0$, and $n_0 = \mu = 1$.

In accordance with (1.8), (1.9) and (1.10) this is equivalent to the situation where the constraints permit the rotation of the system (1) about the moving straight line Az and a translation of the system (2) in the constant direction n', perpendicular to the vertical plane which passes through AC. Point C coincides with B and is fixed in the system of coordinates xyz. Displacements of the systems (1) and (2) are assumed to be the rigid body displacements.

Forces of interaction of system (1) on system (2) yield according to (1.14) and (1.15) a moment equal to zero with respect to the axis parallel to Az and passing through C.

External forces acting on the system (1) yield, according to (1.13) a moment equal to zero with respect to axis Az. External forces acting on the system (2) yield, according to (1.11), (1.12) and (1.13) an arbitrary couple, with moment $\mathbf{M}^{\circ\prime}$ and force $\mathbf{F}'$ located in the plane through A' parallel to straight lines AC and ω_1.

In this case, the integral (1.20) yields

$$\varkappa\lambda' M' \left(a \frac{d\beta'}{dt} - b \frac{d\alpha'}{dt}\right) + S_3 = \text{const}$$

From Equations (1.10) and (1.16) it follows, that $\varkappa = 1$. Equations (1.7) give

$$\lambda' M' \frac{d\alpha'}{dt} = M' \frac{d}{dt}(\alpha' + f'), \qquad \lambda' M' \frac{d\beta'}{dt} = M' \frac{d}{dt}(\beta' + g') \qquad \begin{matrix}(\alpha' + f' = \alpha + f)\\(\beta' + g' = \beta + g)\end{matrix}$$

Here f' and g' are the coordinates of the center of gravity G' of system (2) with respect to the axes $x'y'z'$, while f and g are the coordinates of the center of gravity G' of system (2) relative to the axes xyz.

The integral becomes

$$M' \left(a \frac{dg}{dt} - b \frac{df}{dt}\right) + M' \left(a \frac{d\beta}{dt} - b \frac{d\alpha}{dt}\right) + S_3 = \text{const}$$

i.e. the form of integral of Chaplygin (integral (11), section 4, [3]). This integral occurs for somewhat different forces and constraints [4] than those indicated by Chaplygin.

3. The results obtained for a mechanical system of material points Λ can be generalized in the same sense as discussed by Chaplygin in section 3, at the end of section 4 and in section 5 of his paper [3]. Theorems and integral of kinetic energy of a mechanical system are related to the properties of a group of actual motions.

BIBLIOGRAPHY

1. Kirchhoff, G., Mechanika. Lektsii po matematicheskoi fizike. Lektsiia 5. (Mechanics. Lectures on mathematical physics. Lecture 5), M., Izd. AN SSSR, 1962.

2. Kotel'nikov, A.P., Vintovoe schislenie i nekotorye prilozheniia ego k geometrii i mechanike (Helical calculus and some of its applications to geometry and mechanics). Kazan', 1895.

3. Chaplygin, S.A., O nekotorom vozmozhnom obobshchenii teoremy ploshchadei s primeneniem k zadache o katanii sharov (On some possible generalization of the theorem of areas with application to the problem of rolling spheres). Izbr. tr. po mekhanike i matematike, M., Gostekhizdat, pp. 434-454, 1954.

4. Bogoiavlenskii, A.A., Ob odnom vide obobshchennogo integrala ploshchadei (On one form of generalized area integral). *PMM* Vol. 21, No. 3, pp. 422-423, 1957.

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